Supplement to Chapter 1

REVIEW QUESTIONS

1.1 Describe the meaning of the terms: (a) microstate, (b) microstate space, (c) sampling sequence.

1.2 How is the probability function $P(x)$ defined, in the frequency interpretation of probability?

1.3 In the normalization condition $\mathbf{P}[\Omega] = 1$, what is the set Ω?

1.4 Given a statement $q(x)$, what is the corresponding subset $Q \subset \Omega$?

1.5 How is the probability $\mathbf{P}[Q]$ of an arbitrary subset $Q \subset \Omega$ defined in the frequency interpretation of probability?

1.6 In the measure theory interpretation of probability, what are the requirements for an acceptable probability distribution $\mathbf{P}[A]$?

1.7 How many different sets of n objects can be constructed from a set of N distinct objects?

1.8 How many ways can N distinct objects be grouped into three sets containing n_1, n_2, and n_3 objects? (Naturally, $n_1 + n_2 + n_3 = N$.)

1.9 Calculate the probability of getting n heads in a throw of N coins.

1.10 Calculate the Poisson distribution for the probability of finding exactly n particles within a small subvolume of an ideal gas.

1.11 What is a statistical ensemble, and how is it related to a sampling sequence?

1.12 Given a statistical ensemble, how do we define the probability $\mathbf{P}[A]$ of a subset $A \subset \Omega$?

1.13 If the microstates are points in a three-dimensional space and A is a three-dimensional region, how is the probability $\mathbf{P}[A]$ related to the probability density function $P(x, y, z)$?

1.14 Given a probability distribution $\mathbf{P}[Q]$ for every subset $Q \subset \Omega$ write formulas for: (a) $\mathbf{P}[q_1 \text{ or } q_2]$, (b) $\mathbf{P}[q_1 \text{ and } q_2]$, and $\mathbf{P}[\text{not } q]$, where q_1 and q_2 are statements.

1.15 If q and q' are statements in terms of the sampling sequence, what is the meaning of the conditional probability $\mathbf{P}[q'|q]$?

1.16 How is the conditional probability $\mathbf{P}[q'|q]$ related to the joint probability $\mathbf{P}[q' \text{ and } q]$?

1.17 What does it mean to say that two statements are statistically independent?

1.18 For statistically independent statements, what is the form of $\mathbf{P}[q \text{ and } q']$?

1.19 What is a random variable?

1.20 Suppose N is an integer random variable with a probability distribution $P_N(n)$, and $X(N)$ is some function of N. What is the probabilty distribution for X?

1.21 Suppose $\mathbf{R}$ is a three-dimensional random variable with a probability density $P_{\mathbf{R}}(\mathbf{r})$, and $Y(\mathbf{R})$ is some scalar function of $\mathbf{R}$. What is the probability density for Y?

1.22 If the continuous variable x has the probability density $P(x)$, how are $\bar{x}$ and Δx defined?

1.23 Prove Chebbyshev's inequality, $\mathbf{P}[|x - \bar{x}| > a] \leq (\Delta x/a)^2$

1.24 Suppose $x_1, x_2, \ldots, x_N$ are N independent random variables with uncertainties Δx_1, $\Delta x_2, \ldots, \Delta x_N$. What is the uncertainty in their average, $x = (x_1 + x_2 + \cdots + x_N)/N$?

1.25 Prove the answer you gave in the last question.

1.26 With the same conditions as Question 1.24 and the added conditions that N is a very large number and that $\bar{x}_1 = \cdots = \bar{x}_N = 0$, what is the form of the probability density associated with the random variable x?

EXERCISES

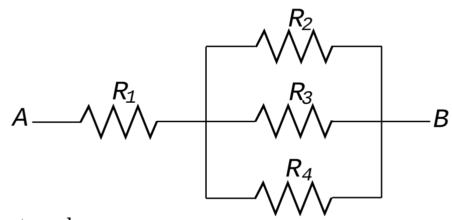

Fig. S1.1 A resistor network.

Exercise 1.1 The probability that resistor R_n $(n = 1, 2, 3, 4)$ is an open circuit $(R_n = \infty)$ is p_n. What is the probability that the resistance between points A and B is infinite?

Solution

$$\begin{aligned} R_{AB} &= R_1 + (R_2^{-1} + R_3^{-1} + R_4^{-1})^{-1} \\ &= R_1 + R' \end{aligned} \tag{S1.1}$$

R' will be infinite if and only if R_2, R_3, and R_4 are all infinite. The probability of that is $p_2p_3p_4$. R_{AB} will be finite only if R_1 and R' are both finite. The probability of that is $(1 - p_1)(1 - p_2p_3p_4)$. Thus

$$\mathbf{P}[R_{AB} = \infty] = 1 - (1 - p_1)(1 - p_2p_3p_4) \tag{S1.2}$$

Exercise 1.2 Let A and B be any two sets. Show that the three sets $S_1 = A$, $S_2 = \overline{A} \cap B$, and $S_3 = \overline{A \cup B}$ are mutually disjoint and that their sum is Ω.

Solution An element x is in S_1 if it is in A. x is in S_2 if it is in B but not in A. (Remember, A and B may intersect.) x is in S_3 if it is in neither A nor B. It is clear that no x can satisfy two of these conditions and that every x must satisfy one of them.

Exercise 1.3 A cube that is painted blue is cut into 64 equal cubes. What is the probability P_n that a little cube, picked at random, has n painted faces, where $n = 0, 1, 2, 3$?

Solution Of the $4^3 = 64$ cubes, there are $2^3 = 8$ inside cubes that have no paint on them. Thus $P_0 = 8/64 = 1/8$. In each of the six faces of the original cube there are 4 small cubes that will end up with one face painted. Thus $P_1 = 6 \times 4/64 = 3/8$. On each of the 12 edges of the original cube there are 2 small cubes that have two painted faces. Thus $P_2 = 12 \times 2/64 = 3/8$. Finally, the 8 corner cubes will have 3 painted faces, so $P_3 = 8/64 = 1/8$. Notice that $P_0 + P_1 + P_2 + P_3 = 1$.

Exercise 1.4 Two faces of a cube, chosen at random, are painted blue. The cube is then cut into 27 equal cubes. What is the probability that one of the little cubes, picked at random, has two blue faces?

Solution The probability that the two painted faces have a common edge is 4/5. This can be seen by imagining that one of the faces has been painted, leaving 5 unpainted faces, 4 of which share an edge with the painted one. If the two pointed faces do not have a common edge, then none of the small cubes will have two blue faces. If they do have a common edge, then 3 of the 27 cubes will have two blue faces. Thus the answer to the question is

$$P = \frac{4}{5}\frac{3}{27} = \frac{4}{45} \tag{S1.3}$$

Exercise 1.5 Assuming that the birth rate is constant throughout the year (which is definitely not true), what is the probability that a person born in this century was born on the 29th of February?

Solution The 29th of February has occurred every 4 years since 1900. The probability that a day, picked at random is the 29th of February is $1/(4 \times 365 + 1) = 0.000684$.

Exercise 1.6 Ten people throw dice, once per minute, at ten tables. When any person throws 12 he leaves. What is the probability that anyone will be left after one hour?

Solution There are 36 possible states for two dice and only one of them gives 12. The probability of *not* hitting 12 on one throw is 35/36. The probability that a given person will be left after one hour is $(35/36)^{60}$. The probability that a given person will be gone after one hour is $1-(35/36)^{60}$. The probability that everyone will be gone after one l hour is $[1-(35/36)^{60}]^{10}$. The answer to the question is

$$\mathbf{P}[\text{someone left}] = 1 - [1-(35/36)^{60}]^{10} = 0.87 \tag{S1.4}$$

Exercise 1.7 A person is given 4 coins, each having equal and independent probabilities of being a penny, a nickel, a dime, or a quarter. What is the probability that the person has been given 37¢?

Solution Imagine that the coins are given one at a time. There are $4^4 = 256$ different possible sequences of four coins. Each has a probability of 1/256. In order to get 37¢, one must have a quarter, a dime, and two pennies. There are 12 ways of arranging those coins. (Imagine that the coins are numbered 1 to 4 as they are given. The quarter can have 4 possible numbers. The dime can then have any of the 3 remaining numbers. The numbers of the pennies are then determined.) Thus the probability of getting 37¢ is

$$P = \frac{12}{256} = \frac{3}{64} \tag{S1.5}$$

Exercise 1.8 If the integer N is chosen at random from a very large interval, what is the probability that the last digit of N^3 is 1?

Solution It is always possible to write N as $N = 10K + d$, where $0 \le d \le 9$ and K is an integer. ($10K$ is what you get by setting the last digit of N to zero.) Then

$$N^3 = K^3 \times 10^3 + 3dK^2 \times 10^2 + 3d^2K \times 10 + d^3 \tag{S1.6}$$

Thus the last digit of N^3 is the same as the last digit of d^3. The last digit of d^3 is 1 only if $d = 1$. Thus the answer to the question is $P = 1/10$.

Exercise 1.9 Given a long table of squares of integers, what is the probability distribution P_n ($n = 0, 1, \ldots, 9$) of the last digits in the table?

Solution By the argument of the previous problem, we can look at the distribution of the last digits in d^2, where $d = 0, 1, \ldots, 9$. But $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, and $9^2 = 81$. Thus $P_0 = 1/10$, $P_1 = 2/10$, $P_2 = 0$, $P_3 = 0$, $P_4 = 2/10$, $P_5 = 1/10$, $P_6 = 2/10$, $P_7 = 0$, $P_8 = 0$, and $P_9 = 2/10$.

Exercise 1.10 What is the probability that a random number from 1 to 1000, inclusive, will be divisible by 13?

Solution $1000/13 = 76 + 12/13$. Thus, there are 76 multiples of 13 between 1 and 1000. Therefore the probability of hitting one of those multiples is $P = 76/1000 = 0.076$. (Note that $1/13 = 0.0769230769230\ldots$.)

Exercise 1.11 N black circles of radius a are printed with their centers placed randomly on a sheet of paper of area A. (Note: The circles are not prevented from overlapping.) Assuming N and A are large enough so that edge effects can be neglected, calculate the average value of the total blackened area.

Solution Because the circles may overlap, the blackened area is less than $N\pi a^2$. Consider an infinitesimal square of area $dA = dx\,dy$ on the paper. dA will be white if the centers of all N circles are further than a from dA. The probability that a particular black circle has its center further than a is $p = (A - \pi a^2)/A = 1 - \pi a^2/A$. The probability that all N centers fall outside the circle of radius a around dA is $P = p^N = (1 - \pi a^2/A)^N$. To write this in a more useful form, let $x = N\pi a^2/A$. Then $P = (1 - x/N)^N$. But $\lim(1 - x/N)^N = e^{-x}$ as $N \to \infty$. Thus the probability that an area dA, picked at random on the sheet, is white is $\exp(-N\pi a^2/A)$. The total blackened area is

$$A' = A\big(1 - e^{-N\pi a^2/A}\big) \tag{S1.7}$$

There are two things to notice about this result.

1. If $N\pi a^2 \ll A$, then we can ignore overlaps and $A' \approx N\pi a^2$.
2. No matter how large N is, there is always a finite probability of being left with a small white area on the sheet. That is $A' < A$.

Exercise 1.12 The integers from 1 to 30 are written on 30 cards. The cards are shuffled and three cards are drawn. What is the probability that the numbers on them are equal to the lengths of the sides of a right triangle?

Solution There is a total of $\binom{30}{3} = 4060$, equally probable, sets of 3 cards. There are basically two integer right triangles, namely (3,4,5) and (5,12,13). Counting multiples of those, the only sets that form right triangles are the 8 sets, (3,4,5), (6,8,10), (9,12,15), (12,16,20), (15,20,25), (18,24,30), (5,12,13), and (10,24,26). The probability of drawing one of those is $8/4060 = 1/508$.

Exercise 1.13 A gas contains a fraction x of isotope A and a fraction $1 - x$ of isotope B. What is the probability that n particles, chosen at random, contain at least one A particle?

Solution The probability that a set of n particles are all B particles is $(1 - x)^n$. The negative of the foregoing statement is the statement that the set contains at least one A particle. Thus the probability of that is $P = 1 - (1 - x)^n$.

Exercise 1.14 In the same situation as was described in the last question, what is the probability that a set of n particles contains at least two A particles?

Solution Let q_0 be the statement that the set of n particles contains no A particles and q_1 be the statement that the set contains exactly one A particle. From the previous answer, we know that $\mathbf{P}[q_0] = (1 - x)^n$. The probability that the first of the n particles is an A and all the rest are B's is $x(1 - x)^{n-1}$. Thus the probability that any one, but only one, is an A particle is $nx(1 - x)^{n-1}$. The set of n particles will contain at least two As if and only if both q_0 and q_1 are false. Noting that q_0 and q_1 are mutually exclusive, we see that

$$\mathbf{P}[\text{at least 2 } A\text{s}] = 1 - (1 - x)^n - nx(1 - x)^{n-1} \tag{S1.8}$$

Exercise 1.15 A random number generator is a computer program that generates a sequence of random real numbers uniformly distributed over the interval from 0 to 1. What is the probability that the product of two numbers from a random number generator is larger than $\frac{1}{2}$?

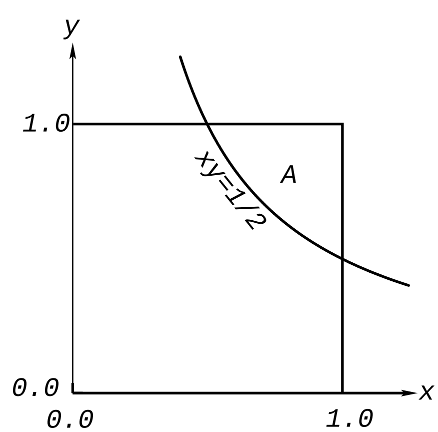

Fig. S1.2 A plot of the function $y = 1/2x$.

Solution Let the two numbers generated be x and y. The point (x, y) is uniformly distributed on the unit square. The points that have a product larger that 1/2 are those that lie in the region A, above the curve

$xy = 1/2$, or equivalently $y = 1/2x$. (See Fig. S1.2.) The area of that region is the integral

$$\begin{aligned} A &= \int_{1/2}^{1} (1 - 1/2x)\, dx \\ &= \tfrac{1}{2} - \tfrac{1}{2}\Big[\log x\Big]_{1/2}^{1} \\ &= \tfrac{1}{2}(1 - \log 2) = 0.153 \end{aligned} \tag{S1.9}$$

which is equal to the probability of getting a pair of numbers whose product is greater than 1/2.

Exercise 1.16 An integer n is drawn at random from the set 1 through 15. Are the statements, $q_1 =$ "n is odd' 'and $q_2 =$ "n is greater than 10" statistically independent?

Solution It is easy to see that $\mathbf{P}[q_1] = 8/15$, $\mathbf{P}[q_2] = 5/15$, and $\mathbf{P}[q_1 \text{ and } q_2] = P(11) + P(13) + P(15) = 3/15$. Thus

$$\mathbf{P}[q_1|q_2] = \frac{\mathbf{P}[q_1 \text{ and } q_2]}{\mathbf{P}[q_2]} = \frac{3/15}{5/15} = \frac{3}{5} \neq \mathbf{P}[q_1] \tag{S1.10}$$

The statements are not statistically independent.

Exercise 1.17 An ensemble consists of a large collection of points in the x–y plane, uniformly distributed within a circle of radius 2, whose center is the origin. Are the statements $q_1 =$ "$x > 1$" and $q_2 =$ "$y > 1$" statistically independent?

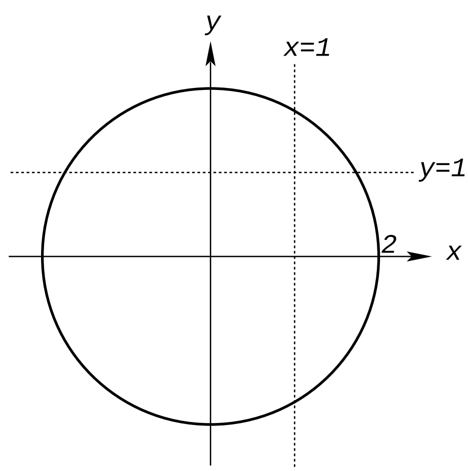

Fig. S1.3

Solution The circle $x^2 + y^2 = 4$ is shown in Fig. S1.3. The area within the circle, lying to the right of the line x=1 we call A_x. That above the line y=1 we call A_y. The equation of the upper boundary of the circle is $y = \sqrt{4 - x^2}$. Therefore, the area A_x shown is given by

$$\begin{aligned} A_x &= 2\int_{1}^{2} \sqrt{4 - x^2}\, dx \\ &= \Big[x\sqrt{4 - x^2} + 4\sin^{-1}\big(\frac{x}{2}\big)\Big]_1^2 \\ &= 4\pi/3 - \sqrt{3} \end{aligned} \tag{S1.11}$$

The area of the intersection, $A' = A_x \cap A_y$ is

$$A' = \int_{1}^{\sqrt{3}} (\sqrt{4 - x^2} - 1)\, dx = \pi/3 - \sqrt{3} + 1 \tag{S1.12}$$

The area of the circle is 4π. Thus

$$\mathbf{P}[q_1] = \mathbf{P}[q_2] = \frac{A_x}{4\pi} \quad \text{and} \quad \mathbf{P}[q_1 \text{ and } q_2] = \frac{A'}{4\pi} \tag{S1.13}$$

It is easy to see that $\mathbf{P}[q_1 \text{ and } q_2] < \mathbf{P}[q_1]\mathbf{P}[q_2]$ and , therefore, q_1 and q_2 are not independent.

Exercise 1.18 A particle moves on the x–y plane. The probability density for its velocity is $P(v_x, v_y)$. From the two assumptions (1) $P(v_x, v_y)$ is a function of $v^2 = v_x^2 + v_y^2$ only, and (2) v_x and v_y are statistically independent, show that $P(v_x, v_y) = (\beta/\pi)\exp(-\beta v^2)$, where β is a constant. This theorem, which can easily be extended to three dimensions, gives a "quick and dirty" derivation of the Maxwell velocity distribution for particles in a gas. It is the one that was originally given by Maxwell. Of course, there is no physical foundation for the second assumption. It does not follow from the first assumption. The first derivation of the Maxwell distribution that was based on physical principles was given by Boltzmann.

Solution From assumption (1), we can write $P(v_x, v_y) = F(v_x^2 + v_y^2)$, where F is some unknown function. From assumption (2), we can write that $P(v_x, v_y) = p(v_x)p(v_y)$, where

$$p(v_x) = \int_{-\infty}^{\infty} P(v_x, v_y)\, dv_y \tag{S1.14}$$

is the probability density for one velocity component. That $p(v_x)$ is equal to $p(v_y)$ follows from assumption (1). Thus

$$p(v_x)p(v_y) = F(v_x^2 + v_y^2) \tag{S1.15}$$

Setting v_x to zero, we get $p(0)p(v_y) = F(v_y^2)$. Using this in Eq. (S1.15) gives

$$F(s + t) = C^2 F(s)F(t) \tag{S1.16}$$

with $s = v_x^2$, $t = v_y^2$, and $C = 1/p(0)$. Differentiating this equation with respect to t and setting t equal to 0, we get

$$F'(s) = C^2 F'(0)F(s) \equiv -\beta F(s) \tag{S1.17}$$

or

$$\frac{d}{ds} \log F(s) = -\beta \tag{S1.18}$$

which has the solution $F(s) = \text{const.} \times e^{-\beta s}$. Therefore

$$P(v_x, v_y) = \text{const.} \times e^{-\beta(v_x^2 + v_y^2)} \tag{S1.19}$$

β cannot be negative because $P(v_x, v_y)$ must be normalizable. The normalization condition for the probability density is

$$\begin{aligned} \int\int P(v_x, v_y)\, dv_x\, dv_y &= \text{const.} \times \left(\int_{-\infty}^{\infty} e^{-\beta v^2} dv \right)^2 \\ &= \text{const.} \times \left(\frac{\pi}{\beta} \right) = 1 \end{aligned} \tag{S1.20}$$

The integral is given in the Table of Integrals. This gives the value of the constant factor and completes the solution.

Exercise 1.19 A one-dimensional gas of noninteracting, statistically uncorrelated particles has an average density of n particles per unit length. What is the probability density function for the distance between any particle and its nearest-neighbor to the right?

Solution Take the location of one of the particles as the origin of our coordinate system. Because the particles are uncorrelated, the fact that there is a particle at the origin does not affect the probability distribution for the other particles. Let $p(x)$ be the probability that there is no particle in the interval from 0 to x. The probability of finding a particle in the interval $(x, x + dx)$ is $n\, dx$. Thus the probability that the *nearest particle* will be in the interval $(x, x+dx)$ is $p(x)\,n\,dx$, which is the answer to our question. We must now calculate $p(x)$. There will be no particle in the interval from 0 to $x+dx$ iff there is no particle in the interval 0 to x and there is no particle in the interval from x to $x+dx$. This means that $p(x+dx) = p(x)(1-n\,dx)$. Using the expansion, $p(x+dx) = p(x) + p'(x)\,dx$, we get the differential equation

$$p'(x) = -n\,p(x) \tag{S1.21}$$

The probability that there is no particle in the interval from 0 to x approaches 1 as x approaches 0. Therefore $p(0) = 1$. With this initial condition, the solution of the differential equation is $p(x) = e^{-nx}$. Therefore, the probability density for the nearest neighbor distance is

$$P(x) = ne^{-nx} \tag{S1.22}$$

Notice that $\int_o^\infty P(x)\,dx = 1$.

Exercise 1.20 In the previous exercise, if we choose a particle at random, what is the probability distribution for the distance to its nearest neighbor, which may be left or right?

Solution Because the particles are independent, the distance to the right neighbor is statistically independent of the distance to the left neighbor. Thus the probability density that the particle's left neighbor is at distance x_L and its right neighbor is at distance x_R is

$$P(x_L, x_R) = n^2 e^{-n(x_L+x_R)} \tag{S1.23}$$

The quantity whose probability distribution we want to calculate is

$$x \equiv \min(x_L, x_R) \tag{S1.24}$$

Using Eq. (1.20), we obtain

$$\begin{aligned} P(x) &= n^2 \int_o^\infty \int_o^\infty \delta(x - \min(x_L, x_R)) e^{-n(x_L+x_R)}\,dx_L\,dx_R \\ &= 2n^2 \int_0^\infty dx_1\,\delta(x-x_1) e^{-nx_1} \int_{x_1}^\infty dx_2\,e^{-nx_2} \\ &= 2n \int_o^\infty dx_1\,\delta(x-x_1) e^{-2nx_1} \\ &= 2ne^{-2nx} \end{aligned} \tag{S1.25}$$

Exercise 1.21 A particle traveling through a dilute gas has a probability of $\gamma\,dt$ of being struck by a gas molecule during any very short time interval dt. What is the probability $P(T)$ that, beginning at time 0, the particle survives for a time interval T without being struck?

Solution The particle will avoid being struck during the time interval $0 < t < T + \Delta T$ iff it is not struck during the time interval $0 < t < T$ and it is not struck during the time interval $T < t < T + \Delta T$. The probability of satisfying the first condition is $P(T)$, and the probability of satisfying the second condition is $1 - \gamma\Delta T$. Therefore

$$P(T+\Delta T) = P(T)(1 - \gamma\,\Delta T) \tag{S1.26}$$

This can be written as

$$\frac{P(T+\Delta T) - P(T)}{\Delta T} = -\gamma P(T) \tag{S1.27}$$

which leads in the limit $\Delta T \to 0$, to the differential equation

$$\frac{dP(T)/dT}{P(T)} = -\gamma \tag{S1.28}$$

The probability that it will survive for a zero time interval is one. That is, $P(0) = 1$. The solution of Eq. (S1.28) that satisfies this initial condition is

$$P(T) = e^{-\gamma T} \tag{S1.29}$$

Exercise 1.22 An ensemble of points in the x–y plane has a probability density $P(x, y) = C \exp(-r^2)$, where $r^2 = x^2 + y^2$. What is the probability that three points, chosen at random from the ensemble, all lie on the same side of the line $x = 1$?

Solution Because $P(x, y) = (\sqrt{C}e^{-x^2})(\sqrt{C}e^{-y^2})$, it is clear that x and y are statistically independent and that the probability distribution for x is $P(x) = \sqrt{C}e^{-x^2}$. In order for $P(x)$ to be normalized, $\sqrt{C}$ must be equal to $1/\sqrt{\pi}$. The probability that a given point will lie to the right of $x = 1$ is given in terms of the complementary error function.

$$\mathbf{P}[x > 1] = \frac{1}{\sqrt{\pi}} \int_1^\infty e^{-x^2} dx = \tfrac{1}{2}\text{erfc}(1) = 0.07865 \tag{S1.30}$$

The probability that a chosen point lies to the left of the line is then

$$\mathbf{P}[x < 1] = 1 - 0.07865 = 0.92135 \tag{S1.31}$$

The probability that three points all lie on the same side is the sum of the probabilities that they all lie to the right of the line and that they all lie to the left of the line.

$$\mathbf{P}[\text{same side}] = (0.07865)^3 + (0.92135)^3 = 0.7826 \tag{S1.32}$$

Exercise 1.23 Two points A and B are chosen at random on the bottom side of a square. Two points A' and B' are chosen at random on the top side. What is the probability that the lines AA' and BB' intersect inside the square? (See Fig. S1.4.)

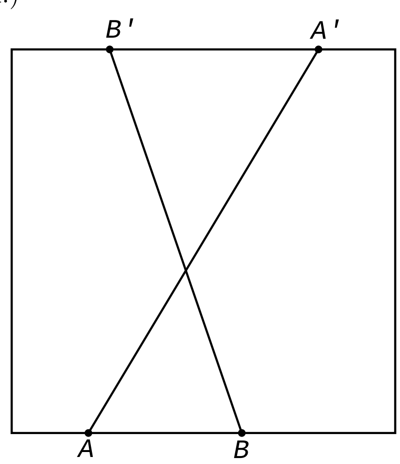

Fig. S1.4 What is the probability that the two lines intersect within the square?

Solution The lines will intersect inside the square iff $A < B$ and $A' > B'$ or $A > B$ and $A' < B'$. The probability of each of the four separate conditions is clearly 1/2. Thus the probability of intersection is

$$P = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 = \tfrac{1}{2} \tag{S1.33}$$

What is the probability that three lines, so drawn, will all intersect?

Exercise 1.24 A particle is distributed along the positive x axis with a probability density $P(x) = a^{-1}e^{-x/a}$. What is the probability density associated with the variable $u = x^3$?

Solution For a given positive number α, the probability that $u < \alpha$ is

$$\begin{aligned}\mathbf{P}[u < \alpha] &= \frac{1}{a}\int_o^{\alpha^{1/3}} e^{-x/a}\,dx \\ &= \int_o^{\alpha^{1/3}/a} e^{-y}\,dy \\ &= \left(1 - e^{-\alpha^{1/3}/a}\right)\end{aligned} \tag{S1.34}$$

But the probability density $P_u(\alpha)$ is the derivative of $\mathbf{P}[u < \alpha]$ with respect to α. Thus

$$P_u(\alpha) = \frac{e^{-\alpha^{1/3}/a}}{3\alpha^{2/3}a} \tag{S1.35}$$

Another way of doing this problem is to use the general formula given in Eq. (1.50). Then

$$P_u(\alpha) = \frac{1}{a}\int_o^{\infty} e^{-x/a}\,\delta(\alpha - x^3)\,dx \tag{S1.36}$$

The general rule for doing an integral involving a delta function of a function is that

$$\int_a^b F(x)\,\delta(\alpha - f(x))\,dx = \sum_r \frac{F(r)}{|f'(r)|} \tag{S1.37}$$

where the sum is over all solutions of the equation $f(x) = \alpha$ that lie in the interval (a, b). In the case at hand, the only solution to the equation $x^3 = \alpha$ in the interval $0 < x < \infty$ is $x = \alpha^{1/3}$, and $f'(x) = d(x^3)/dx = 3x^2$.

Exercise 1.25 The particles in a gas in a uniform gravitational field g at temperature T have a probability density $P(z) = h^{-1}e^{-z/h}$, where the scale height $h = kT/mg$ and k is Boltzmann's constant. If two particles are picked at random, what is the probability density associated with the absolute difference in their heights $Z = |z_1 - z_2|$?

Solution By Eq. 1.50,

$$P_Z(x) = \frac{1}{h^2}\int\int \delta(x - |z_1 - z_2|)e^{-(z_1+z_2)/h}\,dz_1\,dz_2 \tag{S1.38}$$

Using the fact that the integrand is symmetric under exchange of z_1 and z_2, we can restrict the region of integration to $z_1 > z_2$ and multiply the result by 2.

$$P_Z(x) = \frac{2}{h^2}\iint_{z_1>z_2} \delta(x - (z_1 - z_2))e^{-(z_1+z_2)/h}\,dz_1\,dz_2 \tag{S1.39}$$

Changing variables from z_1 to $u = z_1 - z_2$ and using the fact that $z_1 + z_2 = u + 2z_2$, we get

$$\begin{aligned}P_Z(x) &= \frac{2}{h^2}\int_o^{\infty} du \int_o^{\infty} dz_2\,\delta(x - u)e^{-(u+2z_2)/h} \\ &= \frac{e^{-x/h}}{h}\end{aligned} \tag{S1.40}$$

Exercise 1.26 A two-digit number is uniformly distributed from 00 to 99. What is the probability function for the sum of the two digits?

Solution There is one combination that gives a sum of 0. There are 2 that give a sum of 1, 3 that give a sum of 2, 4 that give a sum of 3, ..., 10 that give a sum of 9, 9 that give a sum of 10, 8 that give a sum of 11, ..., 1 that gives a sum of 18. If s is the sum of the digits, then $0 \leq s \leq 18$ and

$$P(s) = \frac{10 - |s-9|}{100} \tag{S1.41}$$

Exercise 1.27 10,000 tickets are numbered from 0000 to 9999. What is the probability of drawing a ticket in which the sum of the first two digits equals the sum of the last two?

Solution The first two digits are statistically independent of the last two digits. Therefore $P(s, s') = P(s)P(s')$. The probability distribution for the sum of either pair of digits was given in the last exercise. The probability that $s = s'$ is

$$\begin{aligned} P = \sum_{s=0}^{18} P(s,s) &= \sum_{s=0}^{18} \frac{(10-|s-9|)^2}{10,000} \\ &= 2\sum_{s=0}^{8} \frac{(1+s)^2}{10,000} + \frac{1}{100} = 0.067 \end{aligned} \tag{S1.42}$$

Exercise 1.28 Let $P_1(x_1)$, $P_2(x_2)$, and $P_3(x_3)$ be the probability densities associated with three independent random variables. If $S = x_1 + x_2 + x_3$, show that

$$P_S(s) = \int P_1(s-v)P_2(v-w)P_3(w)\,dv\,dw \tag{S1.43}$$

Solution We know that

$$P_S(s) = \int P_1(x_1)P_2(x_2)P_3(x_3)\,\delta(s - x_1 - x_2 - x_3)\,dx_1\,dx_2\,dx_3 \tag{S1.44}$$

Introducing variables $u = x_1 + x_2 + x_3$, $v = x_2 + x_3$, and $w = x_3$. The Jacobian of the transformation is

$$\frac{\partial(u,v,w)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \tag{S1.45}$$

The inverse transformation is $x_1 = u - v$, $x_2 = v - w$, and $x_3 = w$. Thus

$$\begin{aligned} P_S(s) &= \int P_1(u-v)P_2(v-w)P_3(w)\delta(s-u)\,du\,dv\,dw \\ &= \int P_1(s-v)P_2(v-w)P_3(w)\,dv\,dw \end{aligned} \tag{S1.46}$$

The integral on the right-hand side is called the *convolution* of the functions P_1, P_2, and P_3. The result can be extended to any number of random variables.

Exercise 1.29 A particle is distributed along the positive x axis with a probability density $P(x) = Cx^2\exp[-(x/a)^2]$. Determine its average coordinate $\bar{x}$ and the fluctuation Δx.

Solution First we must determine the normalization constant C.

$$C\int_o^\infty x^2 e^{-(x/a)^2}\,dx = Ca^3\int_o^\infty u^2 e^{-u^2}\,du = C\frac{a^3\sqrt{\pi}}{4} = 1 \tag{S1.47}$$

Therefore $C = 4/a^3\sqrt{\pi}$. Then

$$\begin{aligned} \bar{x} &= \frac{4}{a^3\sqrt{\pi}}\int_o^\infty x^3 e^{-(x/a)^2}\,dx \\ &= \frac{4a}{\sqrt{\pi}}\int_o^\infty u^3 e^{-u^2}\,du = \frac{2a}{\sqrt{\pi}} \end{aligned} \tag{S1.48}$$

and $(\Delta x)^2 = \overline{x^2} - \bar{x}^2$ where

$$\begin{aligned}\overline{x^2} &= \frac{4}{a^3\sqrt{\pi}} \int_o^\infty x^4 e^{-(x/a)^2} dx \\ &= \frac{4a^2}{\sqrt{\pi}} \int_o^\infty u^4 e^{-u^2} du = \tfrac{3}{2}a^2\end{aligned} \tag{S1.49}$$

Therefore

$$(\Delta x)^2 = (\frac{3}{2} - \frac{4}{\pi})a^2 = 0.227a^2 \tag{S1.50}$$

Exercise 1.30 The coordinate of a particle is distributed over the whole x axis with a probability density $P(x) = C/(a^2 + x^2)^{3/2}$. Calculate C, $\bar{x}$, and Δx.

Solution

$$C^{-1} = \int_{-\infty}^\infty \frac{dx}{(a^2 + x^2)^{3/2}} = \Big[\frac{x}{a^2\sqrt{a^2 + x^2}}\Big]_{-\infty}^\infty = \frac{2}{a^2} \tag{S1.51}$$

$\bar{x} = 0$ because $P(x) = P(-x)$. Therefore

$$(\Delta x)^2 = \overline{x^2} = \frac{a^2}{2} \int_{-\infty}^\infty \frac{x^2\, dx}{(a^2 + x^2)^{3/2}} = \infty \tag{S1.52}$$

For this probability distribution, the mean-square fluctuation diverges!

Exercise 1.31 The Fourier transform of the probability density,

$$\tilde{P}(k) = \int_{-\infty}^\infty e^{ikx} P(x)\, dx \tag{S1.53}$$

is called the *characteristic function* of the random variable x. Let $F(k) = \log \tilde{P}(k)$ and prove that: (1) $F(0) = 0$, (2) $F'(0) = i\bar{x}$, and (3) $F''(0) = -(\Delta x)^2$.

Solution $\tilde{P}(0) = \int P(x)\, dx = 1$. Therefore $F(0) = \log 1 = 0$.

$$\tilde{P}'(0) = i \int x P(x)\, dx = i\bar{x} \tag{S1.54}$$

$$F'(0) = \frac{\tilde{P}'(0)}{\tilde{P}(0)} = i\bar{x} \tag{S1.55}$$

$$\tilde{P}''(0) = -\int x^2 P(x)\, dx = -\overline{x^2} \tag{S1.56}$$

But $F'(k) = \tilde{P}'(k)/\tilde{P}(k)$ and therefore

$$F''(k) = \frac{\tilde{P}''(k)}{\tilde{P}(k)} - \frac{\big[\tilde{P}'(k)\big]^2}{\big[\tilde{P}(k)\big]^2} \tag{S1.57}$$

which implies that

$$F''(0) = -\overline{x^2} + \bar{x}^2 \tag{S1.58}$$

Exercise 1.32 N points are randomly and independently distributed over an interval L. Calculate the probability that some subinterval ℓ will contain exactly n points.

Solution Assume that the N points are laid down one after the other. The probability that a particular point will fall in ℓ is ℓ/L and the probability that it will fall outside ℓ is $(L-\ell)/L$. Therefore, the probability that the first n points will fall inside ℓ and the last $N-n$ points will fall outside ℓ is $\ell^n(L-\ell)^{N-n}/L^N$. The probability that any set of n points will fall in ℓ and all other points outside ℓ is

$$P = \binom{N}{n}\frac{\ell^n(L-\ell)^{N-n}}{L^N} \tag{S1.59}$$

Exercise 1.33 A pair of two-dimensional vector variables, $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, are statistically independent and have probability densities $P_1(\mathbf{x})$ and $P_2(\mathbf{y})$. Let $\mathbf{u} = \mathbf{x} + \mathbf{y} = (u_1, u_2)$ and show that

$$\langle \mathbf{u} \rangle = \langle \mathbf{x} \rangle + \langle \mathbf{y} \rangle \tag{S1.60}$$

and

$$\langle (u_i - \bar{u}_i)(u_j - \bar{u}_j)\rangle = \langle (x_i - \bar{x}_i)(x_j - \bar{x}_j)\rangle + \langle (y_i - \bar{y}_i)(y_j - \bar{y}_j)\rangle \tag{S1.61}$$

where $i, j = 1, 2$. The quantity $\langle (u_i - \bar{u}_i)(u_j - \bar{u}_j)\rangle$ is called the *covariance matrix*. For a one-dimensional variable it has only one component, equal to $(\Delta u)^2$.

Solution Since the variables are statistically independent, their joint probability is $P(\mathbf{x}, \mathbf{y}) = P_1(\mathbf{x})P_2(\mathbf{y})$.

$$\begin{aligned}\langle \mathbf{u} \rangle &= \int (\mathbf{x}+\mathbf{y}) P_1(\mathbf{x})P_2(\mathbf{y})\, d^2x\, d^2y \\ &= \int \mathbf{x} P_1(\mathbf{x})\, d^2x \int P_2(\mathbf{y})\, d^2y + \int P_1(\mathbf{x})\, d^2x \int \mathbf{y} P_2(\mathbf{y})\, d^2y \\ &= \langle \mathbf{x} \rangle + \langle \mathbf{y} \rangle\end{aligned} \tag{S1.62}$$

Also,

$$\langle (u_i - \bar{u}_i)(u_j - \bar{u}_j)\rangle = \langle (x_i - \bar{x}_i + y_i - \bar{y}_i)(x_j - \bar{x}_j + y_j - \bar{y}_j)\rangle \tag{S1.63}$$

But,

$$\langle (x_i - \bar{x}_i)(y_j - \bar{y}_j)\rangle = \int P_1(\mathbf{x})(x_i - \bar{x}_i)\, d^2x \int (y_j - \bar{y}_j) P_2(\mathbf{y})\, d^2y = 0 \tag{S1.64}$$

Similarly $\langle (y_i - \bar{y}_i)(x_j - \bar{x}_j)\rangle = 0$, which easily gives the desired result for the covariance matrix.

Exercise 1.34 Given a random variable x with average $\bar{x}$ and uncertainty Δx and a smooth function $y(x)$, show that, in the case that Δx is very small, if we keep only first-order terms in Δx, then

$$\bar{y} = y(\bar{x}) \quad \text{and} \quad \Delta y = |y'(\bar{x})|\, \Delta x \tag{S1.65}$$

Solution For any particular value of x, let $\delta x \equiv x - \bar{x}$. Then $x = \bar{x} + \delta x$, $\langle \delta x \rangle = 0$, and $(\Delta x)^2 = \langle (\delta x)^2 \rangle$. Using these relations, we see that

$$\bar{y} = \langle y(x) \rangle = \langle y(\bar{x} + \delta x) \rangle \approx y(\bar{x}) + y'(\bar{x})\langle \delta x \rangle + \tfrac{1}{2} y''(\bar{x}) \langle (\Delta x)^2 \rangle \approx y(\bar{x}) \tag{S1.66}$$

and

$$\begin{aligned}(\Delta y)^2 &= \langle [y(\bar{x} + \delta x) - y(\bar{x})]^2 \rangle \\ &\approx \big[y'(\bar{x})\big]^2 \langle (\delta x)^2 \rangle \\ &= \big[y'(\bar{x})\big]^2 (\Delta x)^2\end{aligned} \tag{S1.67}$$

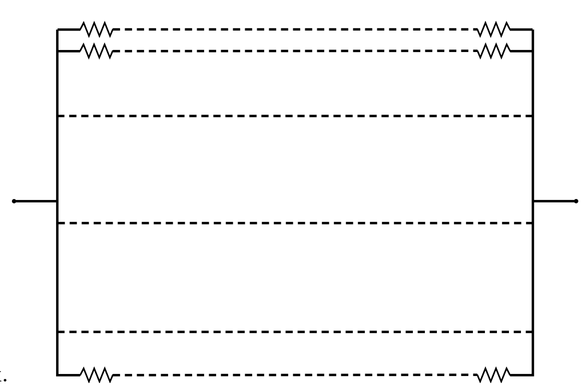

Fig. S1.5 A large resistor network.

Exercise 1.35 The rectangular array of resistors shown in Fig. S1.5 has K parallel rows of L resistors in series. The resistors are drawn from a lot with an average resistance of $1\,\Omega$ and an uncertainty of 10%. Assuming that K and L are large numbers, calculate the average of and the uncertainty in the equivalent resistance of the array.

Solution The single series branches have, according to the law of large numbers, a mean resistance of $L\,\Omega$ and an uncertainty of $\Delta R = 0.1\sqrt{L}\,\Omega$. The conductance of a branch is given by $C = 1/R$. Using the result of the previous exercise, we see that

$$\langle C\rangle = \frac{1}{\langle R\rangle} = L^{-1} \tag{S1.68}$$

and

$$\Delta C = \frac{\Delta R}{\langle R\rangle^2} = 0.1\,L^{-3/2} \tag{S1.69}$$

The conductance of the whole array, C_A, is the sum of the conductances of the branches. Therefore

$$\langle C_A\rangle = K\langle C\rangle = K/L \tag{S1.70}$$

and

$$\Delta C_A = \sqrt{K}\,\Delta C = 0.1\,\sqrt{K}/L^{3/2} \tag{S1.71}$$

The resistance of the array is $R_A = 1/C_A$. Using the result of the previous exercise again, we get

$$\langle R_A\rangle = \frac{L}{K} \quad \text{and} \quad \Delta R_A = \frac{\Delta C_A}{\langle C_A\rangle^2} = 0.1\,L^{1/2}K^{-3/2} \tag{S1.72}$$

Notice that

$$\frac{\Delta R_A}{\langle R_A\rangle} = \frac{0.1}{\sqrt{LK}} \tag{S1.73}$$

which is the uncertainty in one element divided by the square root of the total number of elements, even though R_A is not simply a sum of the resistances of all the elements.

Exercise 1.36 Consider an $N \times N$ array of resistors, similar to that shown in the previous exercise. Assume that each resistor has a probability of 1 % of being an open circuit ($R = \infty$). What is the probability that the whole array is an open circuit? Evaluate the result for $N = 3$ and 1000.

Solution The probability that any particular series branch has a finite resistance is $(0.99)^N$. Therefore, the probability that the branch has an infinite resistance is $1 - (0.99)^N$. The whole circuit will have an infinite resistance only if every branch has an infinite resistance. The probability of that is

$$P = \left[1 - (0.99)^N\right]^N \tag{S1.74}$$

For $N = 3$, $P \approx 0.000026$ and, for $N = 1000$, $P \approx 0.96$.

Exercise 1.37 For a random variable X, with a range of $-\infty < X < \infty$, it is known that

$$\mathbf{P}[X < x] = A + B\tan^{-1}x \tag{S1.75}$$

Determine the values of A and B and the probability density $P_X(x)$.

Solution The limit as $x \to -\infty$ of $\mathbf{P}[X < x]$ must be zero. But $\tan^{-1}(-\infty) = -\pi/2$. Therefore $A = \pi B/2$. The limit as $x \to \infty$ of $\mathbf{P}[X < x]$ must be one. Thus

$$\frac{\pi B}{2} + \frac{\pi B}{2} = 1 \tag{S1.76}$$

or $B = 1/\pi$. Therefore

$$\mathbf{P}[X < x] = \tfrac{1}{2} + \pi^{-1}\tan^{-1}x \tag{S1.77}$$

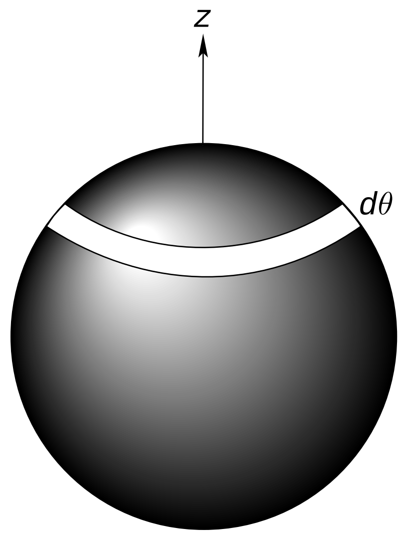

Fig. S1.6 The magnetic moment vector is equally distributed over the sphere.

But

$$P_X(x) = \frac{d}{dx}\mathbf{P}[X < x] = \frac{1}{\pi(1+x^2)} \tag{S1.78}$$

Exercise 1.38 An atom has a magnetic moment that is a vector $\mathbf{m}$ of fixed length m. All possible directions of $\mathbf{m}$ are equally likely. What is the probability density associated with the z component of $\mathbf{m}$?

Solution Picture a very large number, N, of such atoms. If we plot the magnetic moment vectors of all N atoms in a three-dimensional vector space, they will be uniformly distributed on a sphere of radius m (Fig. S1.6). Let $m_z = m\cos\theta$. Clearly, the probability that θ lies in some small interval $d\theta$ is equal to the area on the sphere associated with the angular range $d\theta$ (that is, $2\pi m^2 \sin\theta\, d\theta$) divided by the total area of the sphere, $4\pi m^2$. That is, $dP = \frac{1}{2}\sin\theta\, d\theta$. But $\cos\theta = m_z/m$, and therefore $\sin\theta\, d\theta = dm_z/m$ and $dP = dm_z/2m$. This implies that $P(m_z) = 1/2m$. That is, the component m_z is uniformly distributed in its range, $-m \le m_z \le m$.

Exercise 1.39 A molecule has a finite number K of nondegenerate rotational states of energies E_1, E_2, $\ldots, E_K$. The molecule is bathed in electromagnetic radiation with a continuous spectrum. If the molecule is in state k, then, during the short time interval dt, it has a probability $A(k \to \ell)\, dt$ of absorbing a photon and making a transition to state ℓ and it also has a probability $E(k \to m)\, dt$ of emitting a photon and making a transition from state k to state m. Suppose that, at $t = 0$, the molecule was in state k_o. Construct an explicit equation for calculating $P_n(t)$, the probability that, at time t, the molecule is in state n.

Solution First, let us define a $K \times K$ *transition rate matrix* $T_{k\ell}$ by saying that

$$T_{k\ell} = \begin{cases} E(k \to \ell), & \text{if } E_\ell < E_k \\ 0, & \text{if } k = \ell \\ A(k \to \ell), & \text{if } E_\ell > E_k \end{cases} \tag{S1.79}$$

Then, during the time interval dt, a molecule in state k has a probability $T_{k\ell}\, dt$ of making a transition to state ℓ, where ℓ is any of the other $K - 1$ states. At time $t + dt$ the molecule will be in state k if it was in state k at time t and made no transition during the time interval dt *or* it was in some other state ℓ and made a transition into state k during the time interval dt (we can, for very small dt, neglect the possibility of the molecule's making two or more transitions). The probability that it was in state k at time t and made no transition during the time interval dt is $P_k(t)(1 - \sum_{\ell \ne k} T_{k\ell}\, dt)$. The probability that it was in some other state ℓ at time t and made a transition to state k during the time interval dt is $\sum_{\ell \ne k} P_\ell(t) T_{\ell k}\, dt$. Therefore, the probability that, at time $t + dt$, it will end up in state k is

$$P_k(t+dt) = P_k(t)\Big(1 - \sum_{\ell\neq k} T_{k\ell}\, dt\Big) + \sum_{\ell\neq k} P_\ell(t) T_{\ell k}\, dt \tag{S1.80}$$

This equation can be rearranged to read

$$\frac{P_k(t+dt) - P_k(t)}{dt} = \sum_{\ell\neq k}\Big[P_\ell(t)T_{\ell k} - P_k(t)T_{k\ell}\Big] \tag{S1.81}$$

The limit $dt \to 0$ obviously gives the set of differential equations

$$\frac{dP_k(t)}{dt} = \sum_{\ell\neq k}\Big[P_\ell(t)T_{\ell k} - P_k(t)T_{k\ell}\Big] \tag{S1.82}$$

which must be solved with the initial condition

$$P_k(0) = \begin{cases} 1, & \text{for } k = k_o \\ 0, & \text{otherwise} \end{cases} \tag{S1.83}$$

This equation is called the *master equation* for a statistical process of this sort.

Exercise 1.40 The following question, called the *Monty Hall Problem* after the television game show host, Monty Hall, was correctly answered by the newspaper columnist Marilyn vos Savant in her weekly column. Her solution stimulated thousands of letters, many from professors of mathematics and statistics, that claimed that her answer was incorrect. Here is the problem:

A game show contestant is presented with three closed doors. Behind two of the doors are goats and behind the third is a new car. The contestant chooses a door without opening it. Monty Hall, who knows which door has the car, always goes to one of the other doors and opens it to reveal a goat. The contestant is then given the opportunity of switching to the other unopened door. Is it to the contestant's advantage to switch, to remain with the original choice, or does it make no difference?

Solution Let us call the doors A, B, and C and assume that door A has been chosen and door C has been opened. Knowing that door C has been opened, we are then faced with a problem of conditional probability. After A has been chosen but before any door was opened there were four possible sequences with the following probabilities (we are assuming that, when the contestant chooses the car, Monty Hall chooses one of the remaining two doors randomly)

1. The car is behind A and Monty Hall opens B. $P[A \text{ and } B] = \frac{1}{6}$.
2. The car is behind A and Monty Hall opens C. $P[A \text{ and } C] = \frac{1}{6}$.
3. The car is behind B and Monty Hall opens C. $P[B \text{ and } C] = \frac{1}{3}$.
4. The car is behind C and Monty Hall opens B. $P[C \text{ and } B] = \frac{1}{3}$.

Given that Monty Hall has opened C, what we need to calculate are the two conditional probabilities

$$P[A|C] = \frac{P[A \text{ and } C]}{P[C]} = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$P[B|C] = \frac{P[B \text{ and } C]}{P[C]} = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}$$

The contestant who consistently switches wins two thirds of the time. An easy way of seeing this is to imagine a new game in which there is a second contestant who always gets whatever is behind the two doors that the first contestant did not pick and in which there are no door openings by Monty Hall and no switches allowed. Clearly, the second contestant would get twice as many cars as the first. But the second contestant in this new game would get exactly as many cars as the first contestant in the old game who consistently switched.